§ Levels of interest

Last time Considered $Y_K = GL_2(\mathbb{Q}) \backslash \left(GL_2(\mathbb{A}_f)/K \times \mathcal{H}^{\pm} \right)$

Here, $K \subseteq GL_2(\mathbb{A}_f)$ open compact, the level.

Up to conjugation, $K \subseteq GL_2(\hat{\mathbb{Z}})$ (uses compactness)

Then $\exists\, N \geq 1$ s.th. $K(N) \subseteq K$

(use openness + $GL_2(\hat{\mathbb{Z}}) = \varprojlim_N GL_2(\mathbb{Z}/N) \underset{CRT}{=} \prod_p GL_2(\mathbb{Z}_p)$,

so $\{K(N)\}_{N \geq 1}$ nbhd basis for 1)

$\Rightarrow$ K always defined by congruence conditions:

$$K = \{ \gamma \in GL_2(\hat{\mathbb{Z}}) \;\; [\gamma \bmod N] \in \underbrace{K/K(N)}_{\subseteq GL_2(\mathbb{Z}/N)} \}$$

(This is a tautology.)

Often $K = \prod_p K_p$, $K_p \subseteq GL_2(\mathbb{Q}_p)$ open + compact

$K_p = GL_2(\mathbb{Z}_p)$ almost all p.

Typical choices $K(N) = \{ \gamma \equiv \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \bmod N \}$

$\subseteq K_1(N) = \{ \gamma \equiv \begin{pmatrix} 1 & * \\ & * \end{pmatrix} \bmod N \}$

$\cong K_0(N) = \{ \gamma \equiv \begin{pmatrix} * & * \\ & * \end{pmatrix} \bmod N \}$

These have natural lattice interpretations:

$V = \mathbb{Q}_p^2$, <u>lattice</u> $\Lambda \subset V$ $\underset{\text{def}}{=}$ $\mathbb{Z}_p$-submodule s.th. $\Lambda \cong \mathbb{Z}_p^2$.

Then $GL_2(\mathbb{Q}_p) \curvearrowright \{ \text{lattices } \Lambda \subset V \}$ transitively

(Given Λ, pick basis v_1, v_2. These also form basis of V, so $\exists\, g \in GL_2(\mathbb{Q}_p)$ with $ge_1 = v_1$, $ge_2 = v_2$. Then $g\, \mathbb{Z}_p^2 = \Lambda$.)

$$\Rightarrow \quad GL_2(\mathbb{Q}_p)/GL_2(\mathbb{Z}_p) \xrightarrow{\cong} \{ \text{lattices } \Lambda \subseteq V \}$$
$$g \longmapsto g \cdot \mathbb{Z}_p^2$$

(Follows from $GL_2(\mathbb{Z}_p) = \mathrm{Stab}(\mathbb{Z}_p^2)$.)

<u>Lem</u>

$GL_2(\mathbb{Q}_p)/K(p^n) \xrightarrow{\cong} \{ \Lambda \subset V + \text{basis } v_1, v_2 \text{ of } \Lambda/p^n\Lambda \}$

$GL_2(\mathbb{Q}_p)/K_1(p^n) \xrightarrow{\cong} \{ \Lambda \subset V + \text{non-torsion } v \in \Lambda/p^n\Lambda \}$

$GL_2(\mathbb{Q}_p)/K_0(p^n) \xrightarrow{\cong} \{ \Lambda \subset V + \text{cyclic } C \subset \Lambda/p^n\Lambda \text{ of order } p^n \}$

In all three cases, $GL_2(\mathbb{Q}_p)$ acts transitively on RHS and subgroup K is stabilizer of the "standard datum" $(\mathbb{Z}_p^2,\ e_1, e_2 \bmod p^n\mathbb{Z}_p^2)$ resp. $(\mathbb{Z}_p^2,\ e_1 \bmod p^n\mathbb{Z}_p)$ resp. $(\mathbb{Z}_p^2,\ \langle e_1\rangle)$ $\square$

<u>We obtain</u> E EC / field k, $k = \bar{k}$, char $k \neq p$

$$\mathrm{Isom}_{\mathbb{Z}_p}(\mathbb{Z}_p^2, T_pE)/K(p^n) \xrightarrow{\cong} \{\text{level } p^n\text{-str on } E\}$$
$$\eta \longmapsto (\eta(e_1 \bmod p^n), \eta(e_2 \bmod p^n))$$

$$\text{---}\,\text{''}\,\text{---}/K_1(p^n) \xrightarrow{\cong} \{x \in E[p^n] \text{ of exact order } p^n\}$$
$$\eta \longmapsto \eta(e_1 \bmod p^n)$$

$$\text{---}\,\text{''}\,\text{---}/K_0(p^n) \xrightarrow{\cong} \{C \subseteq E[p^n],\ C \cong \mathbb{Z}/p^n\}$$
$$\eta \longmapsto \langle \eta(e_1 \bmod p^n)\rangle$$

§ An example $N \geq 3$, $p \nmid N$.

$$K(N,p) := K(N) \cap K_0(p) \subseteq GL_2(\hat{\mathbb{Z}})$$

(Motivation $K_0(p)$ is most important level for Hecke operator theory, but always $-1 \in K_0(m)$.
So one cannot get a fine moduli space for level of type $K_0(m)$. Thus we add auxiliary level structure away from prime of interest.)

Note $K(N,p) = \prod_{\ell \mid N} K_\ell(\ell^{v_\ell(N)}) \times K_0(p) \times \prod_{\ell \nmid pN} GL_2(\mathbb{Z}_\ell)$

Lem $Y_{K(N,p)} \cong \{ (E, \alpha, C)/\mathbb{C} \} / \cong$ where

·) $E/\mathbb{C}$ EC

·) $\alpha : \mathbb{Z}/N^2 \xrightarrow{\cong} E[N]$ level-N-str

·) $C \subseteq E[p]$ of order p.

·) $(E, \alpha, C) \cong (E', \alpha', C') \underset{def}{=}$

$\exists \gamma : E \xrightarrow{\cong} E'$ s.th. $\gamma \circ \alpha = \alpha'$, $\gamma(C) = C'$.

<u>Proof</u> $\{(E,\alpha,C)\}/\cong$

$$\xrightarrow{\cong} \; GL_2(\mathbb{Z})\backslash \{(E, (\tau_1,\tau_2), \eta)\}/K(N,p)$$

·) $\tau_1, \tau_2 \in \pi_1(E,e)$ basis

·) $\eta : \hat{\mathbb{Z}}^2 \xrightarrow{\cong} T(E) := \prod_p T_p(E)$ full level str.

·) $GL_2(\mathbb{Z})$ acts as $(\tau_1,\tau_2)\cdot\gamma^t$ (from left)

·) $K = K(N,p)$ acts as $\eta\circ g$. (from right)

·) <u>Map</u>: Pick any τ_1, τ_2, pick η s.th.

$\eta \equiv \alpha \mod N$ and $\langle \eta(e_1) \mod p\rangle = C$.

$$\xrightarrow{\cong} \; GL_2(\mathbb{Z})\backslash\left(GL_2(\hat{\mathbb{Z}})/K \times \mathcal{H}^{\pm}\right)$$

by $(E,(\tau_1,\tau_2),\eta) \longmapsto (\hat{\tau}^{-1}\circ\eta, \tau_1/\tau_2)$

<u>Explanation</u> (τ_1,τ_2) gives full level structure

$\hat{\tau} : \hat{\mathbb{Z}}^2 \to T(E)$, $e_i \mapsto \left(\frac{\tau_i}{n}\right)_{n\geq 1}$

Then η differs from $\hat{\tau}$ by <u>unique</u> $h \in GL_2(\hat{\mathbb{Z}})$:

$$h = (\hat{\tau})^{-1}\circ\eta \qquad \begin{tikzcd} \hat{\mathbb{Z}}^2 \arrow[r, "\eta"] \arrow[d] & T(E) \\ \hat{\mathbb{Z}}^2 \arrow[ur, "\hat{\tau}"'] & \end{tikzcd}$$

This provides a bijection

$$\{(E, (\tau_1,\tau_2), \eta)\}/_{\cong} \;\cong\; \mathrm{GL}_2(\hat{\mathbb{Z}}) \times \mathcal{H}^{\pm}$$

$\mathrm{GL}_2(\mathbb{Z}) \times K$ - action on RHS become

$$\gamma \cdot (h, \tau) \cdot g = ({}^t\gamma^{-1} \cdot h \cdot g, \gamma\tau)$$

$$\xrightarrow{\cong} \mathrm{GL}_2(\mathbb{Q}) \backslash \left(\mathrm{GL}_2(\mathbb{A}_f)/K \times \mathcal{H}^{\pm} \right)$$

by $[h,\tau] \longmapsto [h,\tau]$ $\left(\text{or} \longmapsto [{}^t h^{-1}, \tau] \right)$

Injectivity $(h_1, \tau_1) = (\gamma h_2 g, \gamma \tau_2)$

$h_i \in \mathrm{GL}_2(\hat{\mathbb{Z}})$, $\gamma \in \mathrm{GL}_2(\mathbb{Q})$, $g \in K$

$\Rightarrow \gamma = h_1 g^{-1} h_2^{-1} \in \mathrm{GL}_2(\hat{\mathbb{Z}})$

$\Rightarrow \gamma \in \mathrm{GL}_2(\mathbb{Q}) \cap \mathrm{GL}_2(\hat{\mathbb{Z}}) = \mathrm{GL}_2(\mathbb{Z})$

Surjectivity Given $[h,\tau]$, use class number 1

property $\mathrm{GL}_2(\mathbb{A}_f) = \mathrm{GL}_2(\mathbb{Q}) \cdot \mathrm{GL}_2(\hat{\mathbb{Z}})$

to write $h = \gamma \cdot h_0$. Then $[h,\tau] = [h_0, \gamma^{-1}\tau]$

$\in$ Image. $\square$

§ Integral models $N \geq 3$, $p \nmid N$ as before

$$M_{N,p} : \left(Sch/\mathbb{Z}[N^{-1}] \right)^{op} \longrightarrow Set$$

$$S \longmapsto \{ (E, \alpha, C) \} / \cong$$

- E/S EC
- $\alpha : \underline{\mathbb{Z}/N}^2_S \xrightarrow{\cong} E[N]$ level-N-str
- $C \hookrightarrow E$ closed subgroup scheme, finite loc free rank p / S.
- $(E, \alpha, C) \cong (E', \alpha', C') \underset{\text{def}}{=}$ $\exists\, \gamma : E \xrightarrow{\cong} E'$ s.th. $\alpha' = \gamma \circ \alpha$, $C' = \gamma(C)$.

Comes with forgetful map:

$$M_{N,p} \longrightarrow M_N, \quad (E, \alpha, C) \longmapsto (E, \alpha).$$

__Thm__ $M_{N,p}$ representable by an affine scheme. It is regular and finite flat of degree $p+1$ over M_N.

__Rmk__ Without the finite flat statement the Thm would be useless, e.g. $\mathbb{F}_p \otimes_{\mathbb{Z}} M_{N,p}$ could be $\emptyset$.

Recall that fin loc free rank r group schemes $/ S$ =

$\mathcal{A}/\mathcal{O}_S$ loc free rank r $\mathcal{O}_S$-module

+ $1 : \mathcal{O}_S \to \mathcal{A}$ unit

+ $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ multiplication

+ $1^* : \mathcal{A} \to \mathcal{O}_S$ counit

+ $m^* : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ comultiplication

s.th. Hopf-algebra axioms satisfied.

<u>Prop</u> G/S rank r fin loc free group scheme. Then

$$\mathrm{Sub}_d(G) : (\mathrm{Sch}/S)^{op} \longrightarrow \mathrm{Set}$$
$$T \longmapsto \left\{ H \hookrightarrow G_T \ \begin{array}{l}\text{fin loc free rank } d \\ \text{subgroup sch}\end{array} \right\}$$

is representable by projective S-scheme.

<u>Proof</u> $G = \underline{\mathrm{Spec}}_{\mathcal{O}_S} \mathcal{A}$. Consider relative Grassmanian of rank d quotients of $\mathcal{A}$ $D = \mathrm{Gr}_d(\mathcal{A})$

$$D(u : T \to S) = \left\{ u^* \mathcal{A} \twoheadrightarrow Q \ , \ Q \text{ loc free rank } d / \mathcal{O}_T \right\}$$

$q : D \to S$ projective, locally $\cong G(r,d)_S$.

Consider $q^*\mathcal{A} \to Q$ universal quotient

Kernel $\mathcal{I}$ loc free rank $r-d$ / $\mathcal{O}_D$.

Hopf algebra datum on $q^*\mathcal{A}$ gives rise to various maps:

<u>Unit</u> $1 : \mathcal{O}_D \to q^*\mathcal{A} \to Q$

<u>Multiplication</u>

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}\otimes q^*\mathcal{A} + q^*\mathcal{A}\otimes\mathcal{I} \arrow[r] \arrow[dr, bend right, "f"'] & q^*\mathcal{A}\otimes q^*\mathcal{A} \arrow[r] \arrow[d, "m"] \arrow[dr] & Q\otimes Q \arrow[r] \arrow[d, dotted, "x"] & 0 \\
 & & q^*\mathcal{A} \arrow[r] & Q &
\end{tikzcd}$$

<u>Counit</u>

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I} \arrow[r] \arrow[dr, "g"'] & q^*\mathcal{A} \arrow[r] \arrow[d, "1^*"] & Q \arrow[r] \arrow[dl, dotted, "y"] & 0 \\
 & & \mathcal{O}_D & &
\end{tikzcd}$$

<u>Comultiplication</u>

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I} \arrow[r] \arrow[dr, bend right, "h"'] & q^*\mathcal{A} \arrow[r] \arrow[d, "m^*"] \arrow[dr] & Q \arrow[r] \arrow[d, dotted, "z"] & 0 \\
 & & q^*\mathcal{A}\otimes q^*\mathcal{A} \arrow[r] & Q\otimes Q \arrow[r] & 0
\end{tikzcd}$$

Then $u^*\mathcal{A} \to \mathcal{B} = b^*Q$ defines

closed subscheme $\iff$ $b^*f = 0$
(i.e. x exists)

\+ section $T \to \underline{\mathrm{Spec}}_{\mathcal{O}_T} \mathcal{B}$
$\iff b^*g = 0$ (i.e. y exists)

\+ subgroup $\iff b^*h = 0$ (i.e. z exists)

Thus $\mathrm{Sub}_d(G) = V(f, g, h) \subseteq D$. $\square$

Representability of $\mathcal{M}_{N,p}$:

Consider $(\mathcal{E}, \alpha)/\mathcal{M}_N$ universal EC + level str.
$\mathcal{E}[p] \to \mathcal{M}_N$ is fin. loc. free grp sch of rank p^2.

Then $\mathcal{M}_{N,p} = \mathrm{Sub}_p(\mathcal{E}[p])$. $\square$

Rmk Becomes more complicated for $K_0(p^n)$, $n \geq 2$ because group of order p^n not nec. cyclic.

Examples 0) $G = \underline{\Gamma}_S$ constant

$$\mathrm{Sub}_d(G) = \underline{\mathrm{Sub}_d(\Gamma)}_S \quad \text{constant.}$$

1) $k = \bar{k}$ char $k = p$

Classification of order p subgroups:

$\mathrm{Sub}_p(\alpha_p^{\oplus 2}) = \mathbb{P}^1(k)$

Namely each $\cong \alpha_p$ again and

$\mathrm{Hom}(\alpha_p, \alpha_p^2) = k^2.$

In partic: $\dim \mathrm{Sub}_d(G)$ can be > 0 !

($\dim \mathrm{Gr}(r,d) = d \cdot (r-d)$ very large,

f, g, h roughly $rd(r+d)$ many equations,

$\mathrm{Sub}_d(G)$ can be complicated)

2) E/k EC, $k = \bar{k}$ char $k = p$.

$$\mathrm{Sub}_p(E[p]) = \begin{cases} \{\mathrm{Spec}\, \mathcal{O}_E / \mathfrak{m}_E^p\} & E \text{ supersing} \\ \{\cong \mathbb{Z}/p, \cong \mu_p\} & E \text{ ordinary.} \end{cases}$$

$\Longrightarrow \mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{M}_{N,p} \longrightarrow \mathbb{F}_p \otimes_{\mathbb{Z}} \mathcal{M}_N$ finite + surjective.

3) $k = \bar{k}$, char $k = p$. S/k

$H \in \mathrm{Sub}_p(\mathbb{Z}/p \times \mu_p)(S)$.

$S = S_0 \amalg S_1$ with

$s \in S_0 \iff H(s) = \{0\} \times \mu_p$.

Above S_1, $H \cap (\{1\} \times \mu_p) \subset \mu_p$

defines a section $S \xrightarrow{h} \mu_p$.

Determines H fully because $H \cap (\{i\} \times \mu_p)$

$= i \cdot h(S)$.

Conversely, any $h : S \to \mu_p$ defines subgroup

$$H = \bigcup_{i \in \mathbb{Z}/p} \{i\} \times i \cdot h(S).$$

$\Longrightarrow \mathrm{Sub}_p(\mathbb{Z}/p \times \mu_p) \cong \mathrm{Spec}\, k \amalg \mu_{p,k}$.

In pts of degree $p+1$.